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The Newtonian as well as the special relativistic dynamics are used to study the stability of orbits of a test particle moving around a black hole plus a dipolar halo. The black hole is modeled by either the usual monopole potential or the Paczyński-Wiita pseudo-Newtonian potential. The full general relativistic similar case is also considered. The Poincaré section method and the Lyapunov characteristic exponents show that the orbits for the pseudo-Newtonian potential models are more unstable than the corresponding general relativistic geodesics.

I. INTRODUCTION.

To consider relativistic effects in many body simulations is not a simple task due to the fact that the metric representing their gravitational interaction is far from being known. For the simplest case of two gravitating bodies the metric is known numerically only for few initial conditions and for a limited amount of time [see for instance, Marronetti et al. ([2000])]. Also, assuming that the metric is known, the use of the geodesic equations to determine the trajectory of the bodies represents a quite non trivial problem.

In general, we have three main ways to consider complex systems: a) A full numeric approach with its inherent limitations due to the use of floating point arithmetics and arbitrariness of discretizations of fundamentally continuous functions and variables. Also we have rather unphysical ad hoc assumptions like the introduction of numerical viscosity. b) The use of perturbative methods that are usually employed together with drastic approximations like the mean field approximation for the potentials in many body simulations. These approximations introduce irreversibility in an intrinsic reversible situation. c) The modeling of the problem with simpler equations in which one takes into account a few essential features of the problem. In general, this model can be solved in a more exact form of the two precedent case. But, we have changed the initial problem for a simpler one that may falsify results. In other words, there is not a perfect method to solve a a complex problem. We believe that all of them are valid when the adequate cautions are taken. Furthermore, they are complementary and the usually not proven mathematical or internal consistence of the methods can be independently checked at least for some particular cases.

Due to the weakness of the gravitational field, far from the particles' horizon, the Newtonian gravity is proven to be a reliable description of the gravitational interaction. One can simulate relativistic effects within the Newtonian theory changing the usual potentials to take into account the existence of the horizon. In other words to model relativistic effects via a pseudo-Newtonian potential. These models are simpler enough to describe complex systems that are far beyond of todays knowledge of the full general relativity, e.g., the n-body simulation of the collision of two galaxies to any degree of resolution.

One of the simplest pseudo-Newtonian potentials to describe behavior of test particles moving close to a black hole is the Paczyński and Wiita ([1980]) pseudo-potential,

$$\Phi = -\frac{GM}{R - R_g}. \quad (1)$$

The addition of the term $R_g = 2GM/c^2$ critically changes the particles' trajectory near the source. Some results like the last stable circular orbit are predicted in this model. Other pseudo-Newtonian models can be found in literature, e.g., the one studied by Semerák and Karas ([1999]) to describe rotating black holes, i.e., to approximate the Kerr solution.

We believe that the study of the Paczyński and Wiita (PW) potential in simple albeit nontrivial situations may shed some light into the correctness of the pseudo-potential approach. In particular, in this article, we study integrability and chaos in a system that represents a spherically symmetric source (monopole) surrounded by a dipolar halo (external dipole), that is the simplest mean potential used to describe astrophysical systems restricted to a core and halo, see for instance Binney and Tremaine ([1987]). Different theoretical approaches are used to study this configuration.

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First we use Newton second law to find the motion equations for test particles ($\vec{a} = -\nabla\Phi$) for two different potentials that describe a core plus a dipolar halo system: a) The standard monopole plus external dipole expansion that solves the usual Laplace equation that is totally integrable, see for instance Grammaticos et al. ([1985]), and b) We replace in the former case the monopole term by the PW potential (1). In this case the trajectories are chaotic like in the equivalent full general relativistic system, Vieira and Letelier ([1997]).

We also analyze the equivalent cases using the special relativistic dynamics. We solve the equation $a^\mu = F^\mu$ with $a^\mu = \frac{d^2x^\mu}{d\tau^2} = \gamma \frac{d}{dt} \left(\gamma \frac{dx^\mu}{dt} \right)$ and $F^\mu = \gamma(-\nabla\Phi \cdot \mathbf{v}/c, -\nabla\Phi)$, where $\gamma = (1 - \mathbf{v}^2/c^2)^{-1/2}$, and Φ is taken as in the Newtonian cases. We first use the monopole plus dipole potential that solves the Laplace equation. A phase space analysis shows that the system is stable. Replacing the monopole term by the PW potential we obtain a very unstable system. We also review the equivalent system in general relativity. The geodesic equations for Schwarzschild monopole plus dipolar halo give us chaotic trajectories in the phase space as shown in Vieira and Letelier ([1997]).

In each one of the studied cases we have an integrable Hamiltonian system of equations for the motion of a test particle moving in a spherically symmetric attraction center (standard monopole, PW potential or Schwarzschild metric) that is perturbed by an external dipole term. In all these situations we can apply the KAM (Kolmogorov, Arnold and Moser) theory, see for instance Tabor ([1989]). Since our mass distribution has axial symmetry we are restricted to an effective two-dimensional problem. In the integrable case, in phase space, the orbits of test particles will be confined to a 2-torus. For a constant value of one of the coordinates we obtain a planar section of the phase space. In the integrable case we shall see closed curves for each initial condition, intersections of invariant tori. While in the non-integrable case some tori will be destroyed and the region will be ergodically fulfilled. In order to evaluate the degree of instability of the orbits in each system we also compute the Lyapunov exponents that indicate us how initially close trajectories separate.

II. NEWTONIAN DYNAMICS.

The standard monopole plus external dipole potential in the usual cylindrical coordinates (r, z, ϕ) is

$$\Phi = -\frac{GM}{\sqrt{r^2 + z^2}} + D z, \quad (2)$$

where D is the dipolar strength, G the Newton constant, and M the mass of the attraction center. We use units such that $GM = 1$, furthermore we shall take $c = 1$. From the angular momentum and energy conservation we find that the motion is restricted to the region defined by

$$E^2 - 1 - \frac{L^2}{r^2} - 2\Phi \geq 0. \quad (3)$$

L is the specific angular momentum of the test particle and $E = \sqrt{1 + 2E_{mech}}$, where

$$E_{mech} = \frac{\dot{r}^2 + \dot{z}^2}{2} + \Phi(r, z) + \frac{L^2}{2r^2}$$

is the specific energy. Note that E become imaginary for $E_{mech} < -0.5$ that is the energy of a particle standing on the black hole horizon. The phase space orbits are studied using the Poincaré section method. In Fig.1 we present the surface of section $z = 0$ for the constants: $L = 3.9$, $E = 0.976$, and $D = 2 \times 10^{-4}$. This surface section characterizes an integrable system as expected.

Now we shall replace the monopolar term by the PW pseudo-Newtonian potential, i.e.,

$$\Phi = -\frac{1}{\sqrt{r^2 + z^2} - 2} + D z. \quad (4)$$

Again, the motion of test particles will be restricted to the region that solves (3) with Φ given by (4). In Fig. 2 we present the surface of section $z = 0$. We take the values for the constants as in the preceding case: $L = 3.9$, $E = 0.976$, and $D = 2 \times 10^{-4}$. Contrary to the previous case we observe chaotic orbits in this Poincaré section.

In principle, the use of the special relativistic dynamics should improve the modeling of general relativity with pseudo-Newtonian potentials, see Abramowicz et al. ([1996]). Although, these authors found that the predicted spectra often differ rather substantially from those obtained in the full general relativity context. From the relativistic motion equation we get

$$\frac{d}{dt}(\gamma + \Phi) = 0 \Rightarrow \gamma + \Phi = E, \quad (5)$$

$$\frac{d\theta}{dt} = \frac{L}{\gamma r^2}. \quad (6)$$

By using the above equations and $u^\mu u_\mu = 1$ we obtain

$$(E - \Phi)^2(1 - \dot{r}^2 - \dot{z}^2 - \frac{L^2}{[(E - \Phi)r]^2}) = 1$$

which is used to calculate the region in which the motion is confined. Finally, the motion equations for the variables r and z are,

$$(\Phi - E)\frac{d^2r}{dt^2} = \frac{\partial\Phi}{\partial r}\left(1 - \frac{dr^2}{dt^2}\right) - \frac{\partial\Phi}{\partial z}\frac{dz}{dt}\frac{dr}{dt} - \frac{L^2}{(E - \Phi)r^3}, \quad (7)$$

$$(\Phi - E)\frac{d^2z}{dt^2} = \frac{\partial\Phi}{\partial z}\left(1 - \frac{dz^2}{dt^2}\right) - \frac{\partial\Phi}{\partial r}\frac{dz}{dt}\frac{dr}{dt}. \quad (8)$$

As in the previous section, we start with the usual monopole plus external dipole potential field, i.e., we identify Φ with (2). In Fig. 3 we draw the Poincaré section defined by the plane $z = 0$. The constants are the same of the preceding section, $L = 3.9$, $E = 0.976$, and $D = 2 \times 10^{-4}$. We notice that the tori were preserved in this case, we have stability of orbits. This is an indication of integrability of the system.

Now we start the study of the PW potential plus dipolar halo, i.e., we identify Φ with (4). Unfortunately we cannot confine the orbits by using the constants attributed to all the preceding cases. We put, $L = 4.2$, $E = 0.972$, and $D = 4.2 \times 10^{-4}$. Now the Poincaré section is taken as $z = -5$. The figure in this case, Fig. 4, represents a very chaotic system. We used the same constants to draw another Poincaré section for PW potential plus dipolar halo using Newtonian dynamics. The results are presented in Fig. 5. We see some stable islands in the negative p_r region that cannot be observed Fig. 4. We conclude then that the orbits obtained in the special relativistic context are less stable than the ones obtained with Newton law. The conjugated variables used were dr/dt and r . We made some tests using $dr/d\tau$ and r . The results were qualitatively the same.

IV. GENERAL RELATIVISTIC DYNAMICS.

We start from the axisymmetric line element

$$ds^2 = e^{2\psi(u,v)}dt^2 - e^{-2\psi(u,v)}(u^2 - 1)(1 - v^2)d\phi^2 - e^{2(\gamma(u,v) - \psi(u,v))}(u^2 - v^2)\left(\frac{du^2}{u^2 - 1} + \frac{dv^2}{1 - v^2}\right), \quad (9)$$

in prolate coordinates (t, u, v, ϕ) . The coordinates u and v are related to the usual cylindrical coordinates by $u = (R_+ + R_-)/(2m)$ and $v = (R_+ - R_-)/(2m)$, where $R_\pm = [r^2 + (z \pm m)^2]^{1/2}$ and $m = GM/c^2$. The Schwarzschild monopole plus a dipolar halo is represented by

$$\psi(u, v) = \frac{1}{2}\log\left(\frac{1+u}{1-u}\right) + Duv. \quad (10)$$

Note that taken the limit, $\lim_{c \rightarrow 0} \psi/c^{-2}$, with aid of l'Hôpital rule, we recover (2). To have the right units to take the limit we need to add a c^{-2} factor to D .

The Einstein equations for these class of solutions as well as the corresponding geodesic equations are studied in great detail in Vieira and Letelier ([1999]). Due to the axial symmetry of the metric again the effective geodesic dynamics of the test particles is restricted to a three dimensional "phase space".

The Poincaré section is draw for $v = 0$ (that is equivalent to $z = 0$). In Fig. 6 we present the section for the values of the constants $L = 3.9$, $E = 0.976$, and $D = 2 \times 10^{-4}$. Chaotic orbits may be observed for instance in the region indicated with a rectangle. A zoom of this region is presented in Fig. 7. We can compare Fig. 6 with Fig. 2 and conclude that the orbits obtained via geodesic equation in general relativity are more stable than the ones obtained from the PW potential plus dipolar halo in Newtonian and special relativistic dynamics.

V. LYAPUNOV EXPONENT

We shall study the Lyapunov exponents for the systems above described to better analyze the orbits stability. We shall use the Lyapunov characteristic number (LCN) that is defined as the double limit

$$LCN = \lim_{\substack{\delta_0 \rightarrow 0 \\ t \rightarrow \infty}} \left[\frac{\log(\delta/\delta_0)}{t} \right], \quad (11)$$

where δ_0 and δ are the deviation of two nearby orbits at times 0 and t respectively. We get the largest LCN using the technique suggested by Benettin et al. ([1976])

We start comparing the LCN for orbits in a PW+Dipole system in special relativity and the LCN for orbits in a PW+Dipole in Newtonian theory. The Constants are $L = 4.1$, $E = 0.972$, and $D = 4.1 \times 10^{-4}$. The maximum LCN was obtained around $r = 20$, $z = -5$, and $p_r = -0.04$. Note that the value of p_z is determined by the constants of motion and the value of r , z and p_r . For the relativistic case we get $LCN = (3.2 \pm 0.4) \times 10^{-4}$ while for the Newtonian approach we obtain $LCN = (1.8 \pm 0.4) \times 10^{-4}$. We did some tests for the usual integrable Newtonian monopole plus dipole system and we always obtain for the LCN at least one order of magnitude lower than the precedent cases.

For orbits of test particles in the the full general relativistic monopole plus dipole system and in the Newtonian PW+Dipole system we chose $L = 3.902$, $E = 0.9756$ and $D = 2.0 \times 10^{-4}$. We obtain for orbits in the PW+Dipole system $LCN = (2.0 \pm 0.5) \times 10^{-4}$. This value was obtained for orbits around $r = 7.5$, $z = 0$, and $p_r = 0$. For the general relativistic system the proper time and the coordinate time were tested in the equation (11) and no significant difference was found. The largest LCN was computed around $u = 9.75$, $v = 0$, and $p_u = -0.038$. As before, p_v is fixed by the value of the other variables and the motion constants. We found always $LCN < 5 \times 10^5$. The Lyapunov like coefficients used in general relativistic systems may have different forms as the one suggested by Burd and Tavakol ([1993]) in the study of Bianchi IX systems. However, we have studied a simple system with no singularities besides the black hole where we have a well defined evolution parameter. Hence, in this case no significant difference should be found by using other definition of the Lyapunov coefficients. Furthermore, in the general relativistic system studied we have several natural ways to choose the space variables e.g., the spheroidal (u, v, ψ) and the cylindrical (r, z, ϕ) . We found no significant differences when either system of coordinates are used to describe the orbits of particles moving a few Schwarzschild radii apart from the central black hole.

In summary, the study of Lyapunov coefficients confirms the qualitative analysis of the Poincaré section method, we have that the general relativistic orbits are more stable than the Newtonian and special relativistic ones. The special relativistic orbits are the most unstable.

VI. DISCUSSION

In the Paczyński-Witta potential, the term $-2GM/c^2$ in the denominator of the equation (4) creates a saddle point in the effective potential in Newtonian as well as in special relativistic dynamics. The addition of the dipole term separates the stable and unstable manifold emanating from the hyperbolic fix point as discussed by Letelier and Vieira ([1998]). In this case, as consequence of the Poincaré-Birkhoff theorem, there is an homoclinic web that gives rise to chaotic motion for bounded orbits in phase space, see for instance Tabor ([1989]).

The chaotic orbits encountered in the pseudo-Newtonian plus dipole system agrees with the general relativistic equivalent situation. However, those effects might be distorted in the PW approach because the Poincaré sections as well as the Lyapunov exponents show more unstable orbits. This instability is magnified when the special relativistic dynamics is used. Vokrouhlický and Karas [1998] studied the stability of orbits for particles gravitating around a $1/R$ Newtonian potential with an axisymmetric perturbation. Sridhar and Touma ([1999]) found for the same class of potentials that the instability decreases in orbits closer to the black hole. This result may not be verified when pseudo Newtonian or full general relativistic model are considered. The main difference being the presence of a saddle point in the effective potential near the black hole. Therefore orbits near the core may be more unstable because of this critical point in the effective potential that is a source of instability.

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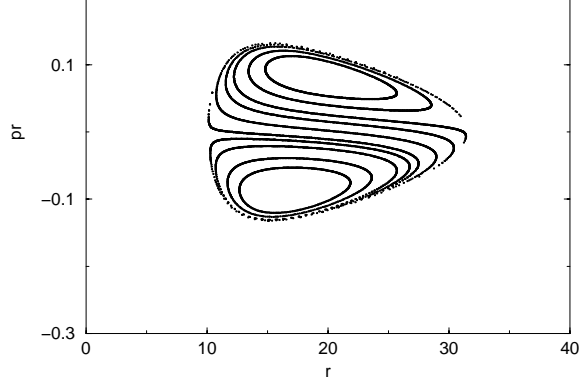


FIG. 1. Surface of section for the Newtonian motion of a test particle in a standard monopole plus external dipole potential for $L_z = 3.9$, $E = 0.976$, and $D = 2 \times 10^{-4}$. The section corresponds to the plane $z = 0$. For these values of the parameters we have the section of an integrable motion.

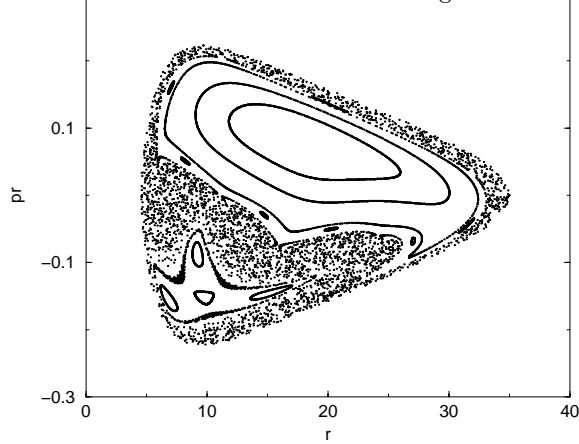


FIG. 2. Surface of section for the Newtonian motion of a test particle in a Paczyński-Wiita potential plus a dipolar halo for $L_z = 3.9$, $E = 0.976$, and $D = 2 \times 10^{-4}$. The section corresponds to the plane $z = 0$. We see chaotic motion.

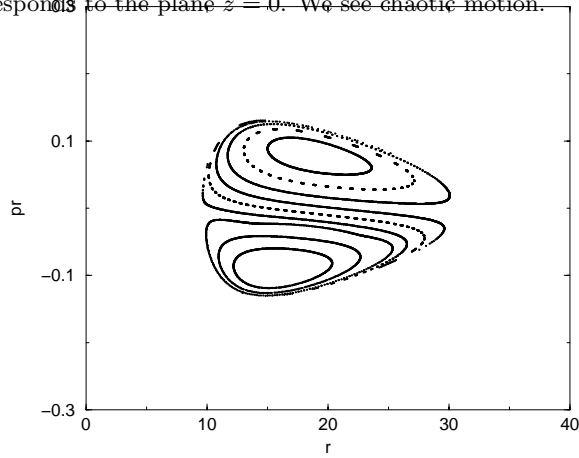


FIG. 3. Surface of section for the special relativistic motion of a test particle in a usual monopole potential plus a dipolar halo for $L_z = 3.9$, $E = 0.976$, and $D = 2 \times 10^{-4}$. The section corresponds to the plane $z = 0$. For these values of the parameters we have the section of a regular motion.

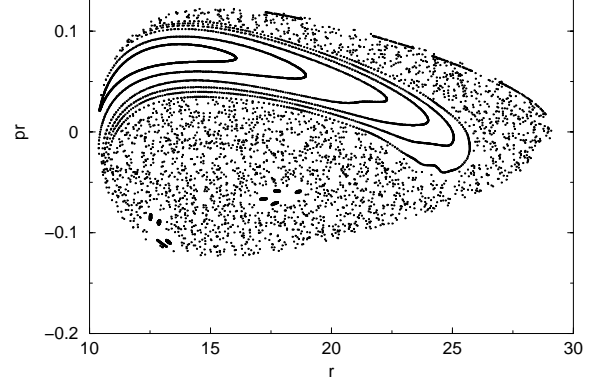


FIG. 4. Surface of section for the special relativistic motion of a test particle in a Paczyński-Wiita potential plus a dipolar halo for $L_z = 4.2$, $E = 0.972$, and $D = 4.2 \times 10^{-4}$. The section corresponds to the plane $z = -5$. We have a very irregular motion.

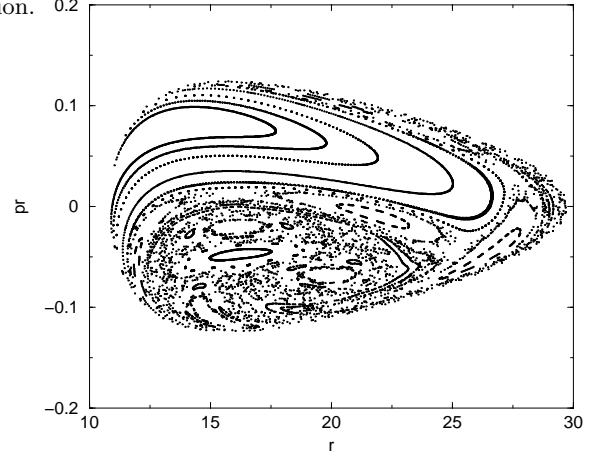


FIG. 5. Surface of section of the Newtonian motion of a test particle in a Paczyński-Wiita potential plus a dipolar halo for $L_z = 4.2$, $E = 0.972$, and $D = 4.2 \times 10^{-4}$. The section corresponds to the plane $z = -5$. We have an irregular motion but it is more stable than the one shown in the precedent figure.

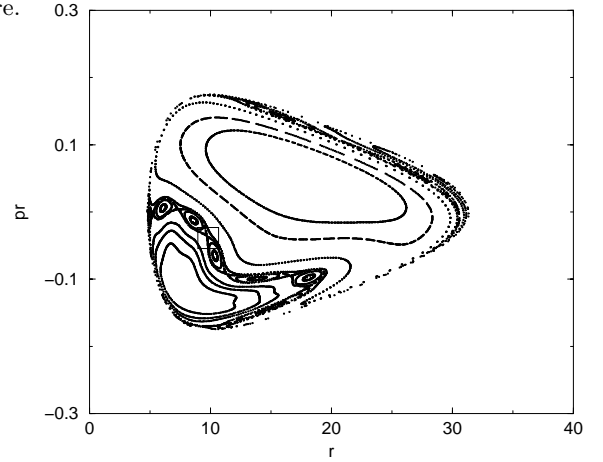


FIG. 6. Surface of section for the geodesic motion of a test particle in a Schwarzschild monopole with a dipolar halo for $L_z = 3.9$, $E = 0.976$, and $D = 2 \times 10^{-4}$. The section corresponds to the plane $v = 0$. For these parameters we have small regions of instability.

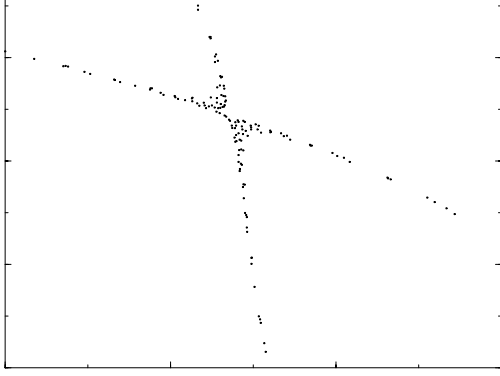


FIG. 7. A zoom of the small boxed region of the previous figure.